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Toward a modified variational iteration method

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Abstract

The variational iteration method (VIM) attracted much attention in the past few years as a promising method for solving nonlinear differential equations. It is shown in this paper that the application of VIM to a special kind of nonlinear differential equations leads to calculation of unneeded terms and more time consumed in repeated calculations for series solutions. A modified VIM is introduced to eliminate the shortcomings; and its effectiveness is illustrated by some examples.

Keywords: Analytical solution; Variational iteration method; Adomian decomposition method; Nonlinear partial differential equation; Burgers' equation; KdV equation

1. Introduction

Ji-Huan He proposed the well-known variational iteration method (VIM) to have a series solution of a nonlinear differential equation using an iterative formula [5]. It was successfully applied to autonomous ordinary differential equations [8], to nonlinear polycrystalline solids [13]. VIM was used to construct solitary solutions and compacton-like solutions for nonlinear dispersive equations [10], to study the regularized long wave equation [14], to solve Burger's and coupled Burger's equations [3], to analyze coupled Schrödinger–KdV, shallow water and generalized KdV equations [15] and other equations, see [7,9] for example.

Insight into the solution procedure of the VIM shows some disadvantages, namely, repeated computations and computations of unneeded terms, which consumes time and effort. A modified variational iteration method (MVIM) is introduced to overcome these disadvantages.

2. Variational iteration method

To illustrate the basic concepts of VIM, consider the following general nonlinear partial differential equation:

$$\begin{aligned} &Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \\ &u(x,0) = f(x), \end{aligned} \tag{1}$$

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where $L=(\partial/\partial t)$, R is a linear operator which has partial derivatives with respect to x, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term.

According to the VIM [9,6], we can construct the following iteration formula:

$$U_{n+1}(x,t)=U_n(x,t)+\int_0^t \lambda\{LU_n+\overbrace{RU_n}^{\sim}+\overbrace{NU_n}^{\sim}-g\}\,\mathrm{d}\tau, \tag{2}$$

where λ is called a general Lagrange multiplier [12] which can be identified optimally via variational theory, $\overbrace{RU_n}^{\sim}$, and $\overbrace{NU_n}^{\sim}$ are considered as restricted variations , i.e. $\delta\,\overbrace{RU_n}^{\sim}=0$, $\delta\,\overbrace{NU_n}^{\sim}=0$.

Calculating variation with respect to U_n, the following stationary conditions are obtained:

$$\lambda'(\tau)=0,$$
$$1+\lambda(\tau)|_{\tau=t}=0. \tag{3}$$

The Lagrange multiplier, therefore, can be identified as $\lambda=-1$.

Substituting the identified multiplier into Eq. (2) results in the following iteration formula:

$$U_{n+1}=U_n-\int_0^t \{L(U_n)+R(U_n)+NU_n-g\}\,\mathrm{d}\tau. \tag{4}$$

The second term on the right is called the correction term. Eq. (4) can be solved iteratively using $U_0(x,t)=f(x)$ as an initial approximation.

2.1. An illustrative example

Consider Burger's equation [11]

$$u_t+uu_x-u_{xx}=0,\quad x\in R, \tag{5}$$

with the constrain

$$u(x,0)=\frac{k}{2}\left(1-\mathrm{Tan}h\left[\frac{kx}{4}\right]\right), \tag{6}$$

where k is a constant.

To solve Eq. (5) by means of the VIM, substitute in Eq. (4) by

$$RU_n=-(U_n)_{xx},$$
$$NU_n=U_n(U_n)_x, \tag{7}$$

and

$$g(x,t)=0,$$

and obtain the following variational iteration formula:

$$U_{n+1}=U_n-\int_0^t \{(U_n)_\tau-(U_n)_{xx}+U_n(U_n)_x\}\,\mathrm{d}\tau. \tag{8}$$

Using (8), the approximate solutions $U_n(x,t)$ are obtained iteratively by substituting:

$$U_0(x,t)=u(x,0)=\frac{k}{2}\left(1-\mathrm{Tan}h\left[\frac{kx}{4}\right]\right). \tag{9}$$

Some approximate solutions are listed below:

$$
\begin{aligned}
U_1 &= U_0 + \frac{k^3}{16}\,\mathrm{Sech}^2\left[\frac{kx}{4}\right] t,\\
U_2 &= U_1 + \frac{k^5}{128}\,\mathrm{Tanh}\left[\frac{kx}{4}\right]\mathrm{Sech}^2\left[\frac{kx}{4}\right] t^2 + \frac{k^7}{1536}\,\mathrm{Tanh}\left[\frac{kx}{4}\right]\mathrm{Sech}^4\left[\frac{kx}{4}\right] t^3,\\
U_3 &= U_1 + \frac{k^5}{128}\,\mathrm{Tanh}\left[\frac{kx}{4}\right]\mathrm{Sech}^2\left[\frac{kx}{4}\right] t^2 + \frac{k^7}{3072}\left(-2 + \mathrm{Cosh}\left[\frac{kx}{2}\right]\right) t^3\\
&\quad + \frac{k^9}{98304}\,\mathrm{Sech}^7\left[\frac{kx}{4}\right]\left(10\,\mathrm{Cosh}\left[\frac{kx}{4}\right] - 5\,\mathrm{Cosh}\left[\frac{3kx}{4}\right] + 16\,\mathrm{Sinh}\left[\frac{kx}{4}\right]\right.\\
&\quad \left. -\ 2\,\mathrm{Sinh}\left[\frac{3kx}{4}\right]\right)\mathrm{Sech}^4\left[\frac{kx}{4}\right] t^4 + \mathrm{O}(t^5),\\
U_4 &= U_1 + \frac{k^5}{128}\,\mathrm{Tanh}\left[\frac{kx}{4}\right]\mathrm{Sech}^2\left[\frac{kx}{4}\right] t^2 + \frac{k^7}{3072}\left(-2 + \mathrm{Cosh}\left[\frac{kx}{2}\right]\right) t^3\\
&\quad + \frac{k^9}{98304}\,\mathrm{Sech}^5\left[\frac{kx}{4}\right]\left(-11\,\mathrm{Sinh}\left[\frac{kx}{4}\right] + \mathrm{Sinh}\left[\frac{2kx}{4}\right]\right) t^4 + \mathrm{O}(t^5),\\
&\ \ \vdots
\end{aligned}
\tag{10}
$$

and so on.

The approximate solution takes the form

$$u(x,t) \simeq U_n(x,t), \tag{11}$$

where n is the final iteration step.

3. Analysis of the solution procedure

From above solution procedure, the approximate solution U_n can be written in the form

$$U_n = B_n^0 + B_n^1 t + B_n^2 t^2 + \cdots + B_n^n t^n + \overset{\sim}{B_n^{n+1}}\, t^{n+1} + \overset{\sim}{B_n^{n+2}}\, t^{n+2} + \mathrm{O}(t^{n+3}), \tag{12}$$

where B_n^m is the coefficient of t^m as $m \leqslant n$ and $\overset{\sim}{B_n^m}$ is the coefficient of t^m as $m > n$. B_n^m is settled and takes the same value for each U_n as $m \leqslant n$. $\overset{\sim}{B_n^m}$ is not settled and does not take the same value for each U_n as $m > n$. For example, the results in (10) can be written as

$$
\begin{aligned}
U_0 &= B_0^0,\\
U_1 &= B_1^0 + B_1^1 t,\\
U_2 &= B_2^0 + B_2^1 t + B_2^2 t^2 + \overset{\sim}{B_2^3}\, t^3,\\
U_3 &= B_3^0 + B_3^1 t + B_3^2 t^2 + B_3^3 t^3 + \overset{\sim}{B_3^4}\, t^4 + \mathrm{O}(t^5),\\
&\ \ \vdots
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
&B_0^0 = B_1^0 = B_2^0 = B_3^0 = \cdots = B_n^0 = \frac{k}{2}\left(1 - \mathrm{Tan}h\left[\frac{kx}{4}\right]\right),\\
&B_1^1 = B_2^1 = B_3^1 = \cdots = B_n^1 = \frac{k^3}{16}\,\mathrm{Sech}^2\left[\frac{kx}{4}\right],\\
&B_2^2 = B_3^2 = \cdots = B_n^2 = \frac{k^5}{128}\,\mathrm{Sech}^2\left[\frac{kx}{4}\right]\mathrm{Tan}h\left[\frac{kx}{4}\right],\\
&\overset{\sim}{B_2^3} = \frac{k^7}{1536}\,\mathrm{Sech}^4\left[\frac{kx}{4}\right]\mathrm{Tan}h\left[\frac{kx}{4}\right],\\
&B_3^3 = \cdots = B_n^3 = \frac{k^7}{3072}\,\mathrm{Sech}^4\left[\frac{kx}{4}\right]\left(-2 + \mathrm{Cos}h\left[\frac{kx}{2}\right]\right),\\
&\overset{\sim}{B_3^4} = \frac{k^9}{98304}\,\mathrm{Sech}^7\left[\frac{kx}{4}\right]\left(10\,\mathrm{Cos}h\left[\frac{kx}{4}\right] - 5\,\mathrm{Cos}h\left[\frac{3kx}{4}\right] + 16\,\mathrm{Sin}h\left[\frac{kx}{4}\right] - 2\,\mathrm{Sin}h\left[\frac{3kx}{4}\right]\right),\\
&\quad\vdots
\end{aligned}
\tag{14}
$$

3.1. Comparison between VIM and Adomian decomposition method

Comparing the results obtained by VIM in the previous illustrative example and that obtained by Adomian decomposition method [2], we find that the VIM components $U_n(x,t)$ have the following relationships with the corresponding Adomian decomposition method components $u_n(x,t)$:

$$
\begin{aligned}
&U_0 = u_0,\\
&U_1 = u_0 + u_1,\\
&U_2 = u_0 + u_1 + u_2 + \overset{\sim}{B_2^3}\,t^3,\\
&U_3 = u_0 + u_1 + u_2 + u_3 + \overset{\sim}{B_3^4}\,t^4 + \mathrm{O}(t^5),\\
&\vdots
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
&u_0(x,t) = \frac{k}{2}\left(1 - \mathrm{Tan}h\left[\frac{kx}{4}\right]\right),\\
&u_1(x,t) = \frac{k^3}{16}\,\mathrm{Sech}^2\left[\frac{kx}{4}\right]t,\\
&u_2(x,t) = \frac{k^5}{128}\,\mathrm{Sech}^2\left[\frac{kx}{4}\right]\mathrm{Tan}h\left[\frac{kx}{4}\right]t^2,\\
&u_3(x,t) = \frac{k^7}{3072}\,\mathrm{Sech}^4\left[\frac{kx}{4}\right]\left(-2 + \mathrm{Cos}h\left[\frac{kx}{2}\right]\right)t^3,\\
&\vdots
\end{aligned}
$$

In the following section, it will be found that $\overset{\sim}{B_n^m}\,t^m$ in variation iteration solution will deteriorate the solution.

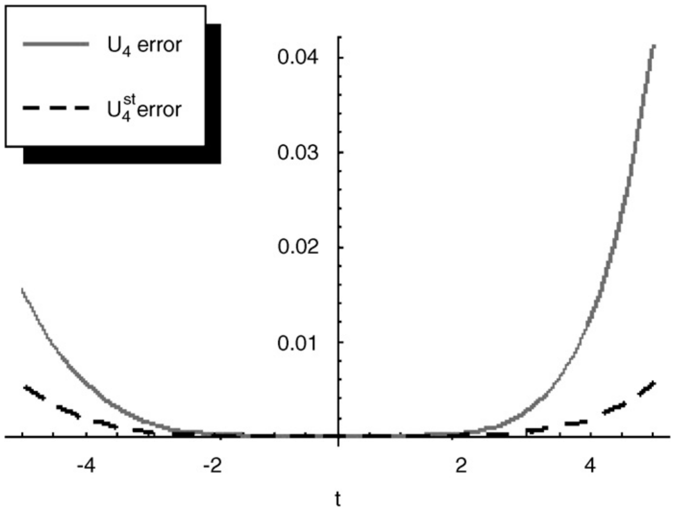

Fig. 1. The absolute error between the exact solution and U_4 and the absolute error between the exact solution and U_4^{st} at $x=0$ and $k=1$.

3.2. Remarks on VIM

Remark 1. Concerning the case of $L=\partial/\partial t$, it can be observed that there are repeated calculations in each step. For example, for $n>0$ the integration $\int_0^t LB_n^0\,\mathrm{d}\tau$, which equals zero, is recalculated in each iteration. Also, the term $B_n^1 t-\int_0^t L(B_n^1\tau)\,\mathrm{d}\tau$, which equals zero too, is recalculated in each iteration step ($n>1$), and so on.

To stop these repeats, the following modification on the recursive formula (4) is suggested:

$$U_{n+1}(x,t)=U_0(x,t)-\int_0^t\{R(U_n(x,\tau))+NU_n(x,\tau)-g(x,\tau)\}\,\mathrm{d}\tau. \tag{16}$$

Via this modification some repeated computations are canceled. Note that the modification may take other forms according to the operator L.

Remark 2. If we re-rewrite Eq. (12) in the form

$$U_n(x,t)=U_n^{\mathrm{st}}(x,t)+U_n^{\mathrm{ns}}(x,t), \tag{17}$$

where U_n^{st} contains the settled terms in Eq. (12) and U_n^{ns} contains the non-settled terms in Eq. (12).

Now, come back again to our previous illustrative example, Fig. 1 shows the absolute error between the exact solution and U_4 and the absolute error between the exact solution and U_4^{st}, where the exact solution of Eq. (5) takes the form

$$u(x,t)=\frac{k}{2}\left(1-\mathrm{Tan}h\left[\frac{k}{4}\left(x-\frac{k}{2}t\right)\right]\right). \tag{18}$$

It can be observed that the addition of the term $U_n^{\mathrm{ns}}(x,t)$ deteriorates the convergence to the exact solution since the coefficients of t^s in $U_n^{\mathrm{ns}}(x,t)$ are not the exact coefficients of t^s. So $U_n^{\mathrm{ns}}(x,t)$ may or may not lead to faster convergence in general. In the illustrated case it slows the convergence down and should be canceled.

To overcome this problem and eliminate $U_n^{\mathrm{ns}}(x,t)$ in the calculated $U_n(x,t)$, the following modification on the recursive formula (16) is suggested:

$$U_{n+1}(x,t)=U_0(x,t)-\int_0^t\{R(U_n(x,\tau))+G_n(x,\tau)-g(x,\tau)\}\,\mathrm{d}\tau, \tag{19}$$

where $U_0=f(x)$ and $G_n(x,t)$ are obtained from

$$NU_n(x,t)=G_n(x,t)+\mathrm{O}(t^{n+1}). \tag{20}$$

Remark 3. It can be observed that not all the repeated calculations are canceled in formula (19), which all the calculations done in U_{n-1} are repeated in calculating U_n.

To stop this repeated computations; let us rewrite Eq. (19) in the following iteration formula:

$$U_{n+1} = U_0 - \int_0^t \{R(U_{n-1}) + G_{n-1} - g\}\,\mathrm{d}\tau - \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}\,\mathrm{d}\tau. \tag{21}$$

But it is known from (19) that

$$U_n = U_0 - \int_0^t \{R(U_{n-1}) + G_{n-1} - g\}\,\mathrm{d}\tau. \tag{22}$$

Substituting by (22) in (21), we get

$$U_{n+1} = U_n - \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}\,\mathrm{d}\tau, \tag{23}$$

where $U_{-1} = 0$, $U_0 = f(x)$, $U_1 = U_0 - \int_0^t \{R(U_0 - U_{-1}) + (G_0 - G_{-1}) - g\}\,\mathrm{d}\tau$, and $G_n(x,t)$ is obtained from

$$NU_n(x,t) = G_n(x,t) + \mathrm{O}(t^{n+1}). \tag{24}$$

This final modified formula (23) cancels all the repeated calculations and terms, which are not needed.

Now if we resolve Eq. (5) with the given initial condition using the modification in (23), with

$$U_{-1} = 0,$$

$$U_0 = \frac{k}{2}\left(1 - \mathrm{Tan}h\left[\frac{kx}{4}\right]\right),$$

and $G_n(x,t)$ is calculated from the relation

$$(U_n(x,t))(U_n(x,t))_x = G_n(x,t) + \mathrm{O}(t^{n+1}). \tag{25}$$

The following modified VIM results are obtained

$$\begin{aligned}
U_0 &= \frac{k}{2}\left(1 - \mathrm{Tan}h\left[\frac{kx}{4}\right]\right),\\
U_1 &= U_0 + \frac{k^3}{16}\mathrm{Sec}h^2\left[\frac{kx}{4}\right]t,\\
U_2 &= U_1 + \frac{k^5}{128}\mathrm{Sec}h^2\left[\frac{kx}{4}\right]\mathrm{Tan}h\left[\frac{kx}{4}\right]t^2,\\
U_3 &= U_2 + \frac{k^7}{3072}\mathrm{Sec}h^4\left[\frac{kx}{4}\right]\left(-2 + \mathrm{Cos}h\left[\frac{kx}{2}\right]\right)t^3\\
U_4 &= U_3 + \frac{k^9}{98304}\mathrm{Sec}h^5\left[\frac{kx}{4}\right]\left(-11\,\mathrm{Sin}h\left[\frac{kx}{4}\right] + \mathrm{Sin}h\left[\frac{2kx}{4}\right]\right)t^4,\\
U_5 &= U_4 + \frac{k^{11}}{3932160}\mathrm{Sec}h^6\left[\frac{kx}{4}\right]\left(33 - 26\,\mathrm{Cos}h\left[\frac{kx}{2}\right] + \mathrm{Cos}h[kx]\right)t^5,\\
&\ \ \vdots
\end{aligned} \tag{26}$$

Table 1
The time consumed in calculating $U_n(x,t)$ using Mathematica-4 package

	Burgers'		KdV		Lax's seventh order KdV	
	VIM	MVIM	VIM	MVIM	VIM	MVIM
$U_1(x,t)$	0.015	0.063	0.016	0.031	0.125	0.109
$U_2(x,t)$	0.141	0.093	0.297	0.094	6.25	0.422
$U_3(x,t)$	3.844	0.094	4.5	0.141	163.032	2.75
$U_4(x,t)$	15.906	0.156	15.25	0.203	N/A	3.125
$U_5(x,t)$	84.75	0.188	128.5	0.297	N/A	4.75
$U_6(x,t)$	1038.86	0.328	673.313	0.422	N/A	7.125

Note: The time results shown in the table are personal and relative and are made just for comparisons.

As predicted, the unneeded terms are omitted and the computations are faster than using VIM alone. Table 1 shows how modified VIM saves time and calculations.

This approximate solution is convergent to the exact solution (18) in a restricted region even if doing more iterative steps.

3.3. Notes on VIM

From the previous analysis and discussion, we can observe that:

1. VIM can obtain a series solution not exactly like Adomian decomposition method.
2. VIM series solution consists of two parts. The first part is the settled part in the approximate solution, which contains the coefficients $B_n^m, m \leqslant n$, and we can depend on it. The second part is the unsettled part in the approximate solution, which contains the coefficients $\widetilde{B}_n^m, m > n$ and we cannot depend on it.
3. VIM needs some modifications to overcome the wasted time in the repeated calculations and the calculations of unneeded terms, namely the unsettled part.

To overcome these disadvantages of VIM, the following MVIM is suggested.

4. The proposed MVIM

Concerning the following partial differential equation,

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t),$$
$$u(x,0) = f(x), \tag{27}$$

where $L = \partial/\partial t$, R is a linear operator which has partial derivatives with respect to x, $Nu(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term. Partial differential equation (27) covers a large branch of applications such as soliton equations like Burgers', coupled Burgers', Schrödinger, KdV, modified KdV and also compacton equations like $k(n,n)$ and many others important equations.

Following the same procedure as done in VIM and using the following iteration formula instead of the iteration formula (4):

$$U_{n+1} = U_n - \int_0^t \{R(U_n - U_{n-1}) + (G_n - G_{n-1})\}\,\mathrm{d}\tau, \tag{28}$$

where $U_{-1} = 0$, $U_0 = f(x)$, $U_1 = U_0 - \int_0^t \{R(U_0 - U_{-1}) + (G_0 - G_{-1}) - g\}\,\mathrm{d}\tau$, and $G_n(x,t)$ is obtained from

$$NU_n(x,t) = G_n(x,t) + \mathrm{O}(t^{n+1}). \tag{29}$$

Eq. (28) can be solved iteratively to obtain an approximate solution that takes the form

$$u(x,t) \simeq U_n(x,t), \tag{30}$$

where n is the final iteration step.

5. Illustrative case studies

To demonstrate the efficiency of MVIM, some selected cases are solved using MVIM:
Case-study 1: Consider the KdV equation which takes the form

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} = 0, \quad x \in R,\\ &u(x,0) = \frac{-k^2}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}x\right]. \end{aligned} \tag{31}$$

Applying VIM, the following VIM results are obtained:

$$\begin{aligned} U_0(x,t) &= \frac{-k^2}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}x\right],\\ U_1(x,t) &= U_0(x,t) - \frac{k^5}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}x\right]\mathrm{Tanh}\left[\frac{k}{2}x\right]t,\\ U_2(x,t) &= U_1(x,t) + \frac{k^8}{8}\,\mathrm{Sech}^4\left[\frac{k}{2}x\right](2 - \mathrm{Cosh}\,[kx])t^2\\ &\quad + 512\,\mathrm{Sech}^6\left[\frac{k}{2}x\right]\mathrm{Tanh}\left[\frac{k}{2}x\right](2 - \mathrm{Cosh}\,[kx])t^3,\\ U_3(x,t) &= U_1(x,t) + \frac{k^8}{8}\,\mathrm{Sech}^4\left[\frac{k}{2}x\right](2 - \mathrm{Cosh}\,[kx])t^2\\ &\quad + \frac{k^{11}}{48}\,\mathrm{Sech}^5\left[\frac{k}{2}x\right]\left(11\,\mathrm{Sinh}\left[\frac{k}{2}x\right] - \mathrm{Sinh}\left[\frac{3k}{2}x\right]\right)t^3 + 64(970\\ &\quad - 1163\,\mathrm{Cosh}[kx] + 232\,\mathrm{Cosh}[2kx] - 11\,\mathrm{Cosh}[3kx])\,\mathrm{Sech}^{10}\left[\frac{k}{2}x\right]t^4 + \mathrm{O}(t^5),\\ &\quad \vdots \end{aligned} \tag{32}$$

Applying MVIM using formula (28) with

$$U_{-1} = 0,$$

$$U_0 = \frac{-k^2}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}x\right],$$

and $G_n(x,t)$ is calculated from the relation

$$-6U_n(x,t)(U_n(x,t))_x = G_n(x,t) + \mathrm{O}(t^{n+1}). \tag{33}$$

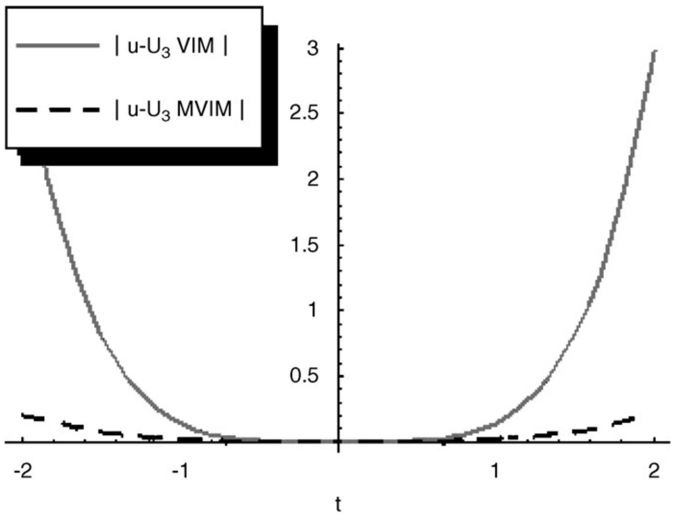

Fig. 2. The absolute error between the exact solution of KdV equation and U_3 generated from VIM and MVIM at $x = 0$ and $k = 1$.

the following MVIM results are obtained:

$$\begin{aligned}
U_0(x,t) &= \frac{-k^2}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}x\right],\\
U_1(x,t) &= U_0(x,t) - \frac{k^5}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}x\right]\mathrm{Tanh}\left[\frac{k}{2}x\right]t,\\
U_2(x,t) &= U_1(x,t) + \frac{k^8}{8}\,\mathrm{Sech}^4\left[\frac{k}{2}x\right](2-\mathrm{Cosh}\,[kx])t^2,\\
U_3(x,t) &= U_2(x,t) + \frac{k^{11}}{48}\,\mathrm{Sech}^5\left[\frac{k}{2}x\right]\left(11\,\mathrm{Sinh}\left[\frac{k}{2}x\right] - \mathrm{Sinh}\left[\frac{3k}{2}x\right]\right)t^3,\\
&\vdots
\end{aligned} \tag{34}$$

As seen, MVIM eliminates all the unneeded terms.

This solution is convergent to the exact solution [1]

$$u(x,t) = \frac{-k^2}{2}\,\mathrm{Sech}^2\left[\frac{k}{2}(x-k^2t)\right]. \tag{35}$$

Fig. 2 shows the absolute error between the exact solution of KdV equation and U_3 generated from VIM and MVIM separately. Table 1 shows how much MVIM saves time and calculations.

Case-Study 2: Consider the Lax's seventh order KdV equation, which takes the form

$$\begin{aligned}
&u_t + (35u^4 + 70(u^2u_{xx} + uu_x^2) + 7(2uu_{xxxx} + 3u_{xx}^2 + 4u_xu_{xxx}) + u_{xxxxxx})_x = 0,\\
&u(x,0) = 2k^2\,\mathrm{Sech}^2\,[kx]\,.
\end{aligned} \tag{36}$$

Applying MVIM using formula (28) with

$$\begin{aligned}
&U_{-1} = 0,\\
&U_0 = 2k^2\,\mathrm{Sech}^2[kx],
\end{aligned}$$

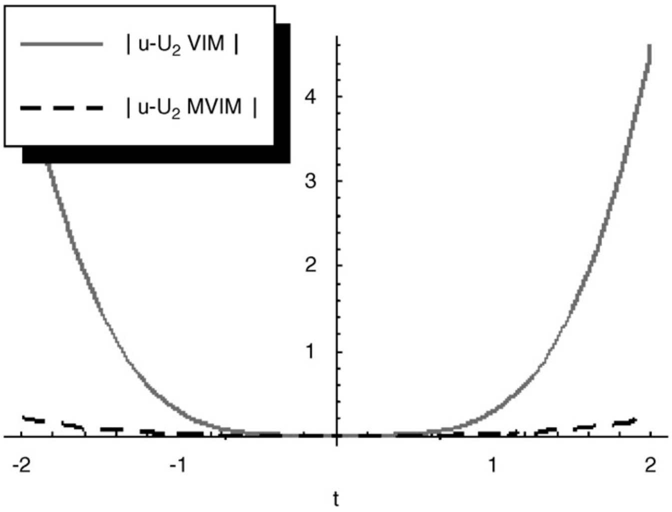

Fig. 3. The absolute error between the exact solution of Lax's seventh order KdV equation and U_2 generated from VIM and MVIM at $x=0$ and $k=0.5$.

and $G_n(x,t)$ is calculated from the relation

$$\begin{aligned}&(35(U_n(x,t))^4+70((U_n(x,t))^2(U_n(x,t))_{xx}+(U_n(x,t))(U_n(x,t))_x^2)\\&\quad+7(2(U_n(x,t))(U_n(x,t))_{xxxx}+3(U_n(x,t))_{xx}^2+4(U_n(x,t))_x(U_n(x,t))_{xxx})\\&\qquad=G_n(x,t)+\mathrm{O}(t^{n+1}),\end{aligned}\tag{37}$$

the following MVIM results are obtained:

$$\begin{aligned}&U_0(x,t)=2k^2\,\mathrm{Sech}^2[kx],\\&U_1(x,t)=U_0(x,t)+256k^9\,\mathrm{Sech}^2[kx]\,\mathrm{Tan}h[kx]t,\\&U_2(x,t)=U_1(x,t)-8192k^{16}\,\mathrm{Sech}^4[kx](2-\,\mathrm{Cos}h[2kx])t^2,\\&U_3(x,t)=U_2(x,t)-\frac{524288}{3}k^{23}\,\mathrm{Sech}^5[kx](11\,\mathrm{Sin}h[kx]-\,\mathrm{Sin}h\,[3kx])t^3,\\&U_4(x,t)=U_3(x,t)+\frac{8388608}{3}k^{30}\,\mathrm{Sech}^6[kx](33-26\,\mathrm{Cos}h[2kx]+\,\mathrm{Cos}h[4kx])t^4,\\&\vdots\end{aligned}\tag{38}$$

This solution is convergent to the exact solution [5]

$$u(x,t)=2k^2\,\mathrm{Sech}^2[k(x-64k^6t)].\tag{39}$$

Fig. 3 shows the absolute error between the exact solution of Lax's seventh order KdV equation and U_2 generated from VIM and MVIM separately at $x=0$ and $k=0.5$. Table 1 shows how much MVIM saves time and calculations.

6. Advantages of MVIM

We can summarize the advantages of MVIM in the following points:

1. MVIM eliminates all the unneeded terms and the repeated computations in VIM.
2. MVIM is powerful in saving time and calculations, see Table 1.

3. Comparing MVIM solution steps in the previous illustrative examples, especially Lax's seventh order KdV equation, with ADM solution steps used in [4], it can be noticed that MVIM is more efficient than ADM. In ADM, Adomian polynomials of six nonlinear terms must be calculated. In MVIM, it is not needed to calculate these polynomials at all.
4. MVIM is often useful to engineering and non-specialists and others to have an approximate closed form solution to describe the nonlinear problems. MVIM can deal with highly nonlinear differential equations with no need to small parameter or linearization. The solution procedure is very simple by means of variational iteration theory, and few iterations lead to high accurate solutions.

7. Conclusion

Some remarks have been noticed on VIM, namely repeated computations and the calculation of unneeded terms. In this paper, we overcome the VIM disadvantages by introducing the MVIM, which stops the repeats of the old computations and eliminate the unneeded terms, which deteriorate the convergence of the approximate solution.

References

[1] T.A. Abassy, M. El-Tawil, H. Kamel, The solution of KdV and mKdV equations using Adomian Padé approximation, Int. J. Nonlinear Sci. Numer. Simulation 5 (4) (2004) 327–339.
[2] T.A. Abassy, M. El-Tawil, H. Kamel, The solution of Burgers' and good Boussinesq equations using ADM–Padé technique, Chaos, Solitons & Fractals, in press.
[3] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181 (2) (2005) 245–251.
[4] S.M. EL-Sayed, D. Kaya, An application of the ADM to seven-order Sawada–Kotara equations, Appl. Math. Comput. 157 (2004) 93–101.
[5] J.H. He, A new approach to nonlinear partial differential equations, Commun. Nonlinear Sci. Numer. Simulation 2 (4) (1997) 230–235.
[6] J.H. He, Variational iteration method—a kind of non-linear analytical technique: some examples, Int. J. Non-Linear Mech. 34 (4) (1999) 699–708.
[7] J.H. He, Approximate analytical solution of Blasius' equation, Commun. Nonlinear Sci. Numer. Simulation 4 (1) (1999) 75–78.
[8] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2–3) (2000) 115–123.
[9] J.H. He, Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B 20 (10) (2006) 1141–1199.
[10] J.H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, Chaos, Solitons & Fractals 29 (1) (2006) 108–113.
[11] E. Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950) 201–230.
[12] M. Inokuti, H. Sekine, T. Mura, General use of the Lagrange multiplier in non-linear mathematical physics, in: S. Nemat-Nasser (Ed.), Variational Method in the Mechanics of Solids, Pergamon Press, Oxford, 1978, pp. 156–162.
[13] V. Marinca, An approximate solution for one-dimensional weakly nonlinear oscillations, Int. J. Nonlinear Sci. Numer. Simulation 3 (2002) 107–120.
[14] A.A. Soliman, Numerical simulation of the generalized regularized long wave equation by He's variational iteration method, Math. Comput. Simulation 70 (2) (2005) 119–124.
[15] A.A. Soliman, A numerical simulation and explicit solutions of KdV-Burgers' and Lax's seventh-order KdV equations, Chaos, Solitons & Fractals 29 (2) (2006) 294–302.